PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. Consider the game below in Figure 1. The first payoff is that of player 1, the second that of player 2.

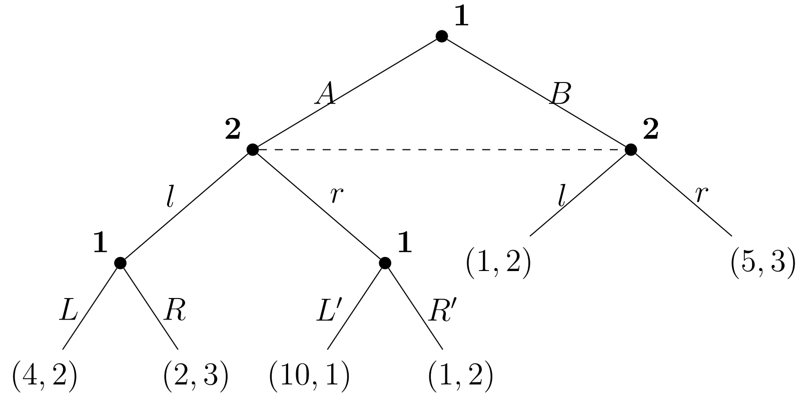


Figure 1: Dynamic 2-Player Game

(a) Is this a game of perfect or imperfect information? How many strategies does each player have? How many proper subgames are there (not including the game itself)?

> **Solution:** Imperfect information. Player 1 has $2 \times 2 \times 2 = 8$ strategies, player 2 has 2 strategies. There are 2 proper subgames.

(b) Find the set of pure-strategy subgame-perfect Nash equilibria (SPNE) of the game.

> **Solution:** In the two proper subgames, the NE is for player 1 play L and L', respectively. Imposing this, the game can be rewritten as the following bimatrix

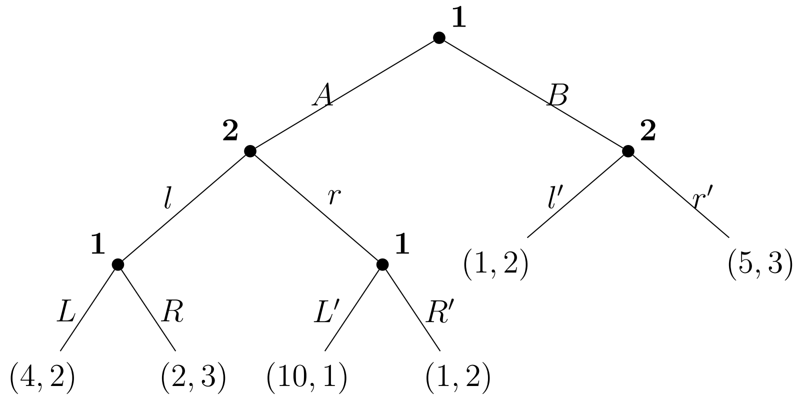
Player 2

Player 1	l	r
A	$\underline{4}, \underline{2}$	$\underline{10}, 1$
B	$1, 2$	$5, \underline{3}$

The unique pure-strategy SPNE is therefore (ALL', l).

(c) Now suppose that player 2 observes player 1's choice of A or B. Draw the game tree of the modified game, and find the set of pure-strategy SPNE.

> **Solution:** The new game tree is depicted below.



The game is now one of perfect information. Player 1 will still prefer L and L'. This will lead player 2 to prefer l and r'. Thus, player 1 will prefer B. The unique pure-strategy SPNE is therefore (BLL', lr').

(d) Compare the outcome and payoffs of the SPNE you found in the original game with the outcome and payoffs of the SPNE in the modified game. If they are the same, comment on why this is. If they are different, explain what causes the difference.

Solution: The outcomes are different. In the modified (perfect information) game, both players achieve a higher SPNE payoff than in the original (imperfect information) game. The reason is that whereas both players prefer the outcome (B, r') over (A, l, L) (here I am just indicating the path in the tree), then player 1 cannot commit to playing B when player 2 plays r in the imperfect information game. He will always deviate to A. Removing the imperfect information removes this commitment problem.

(e) Can you change one payoff (but just one) of one of the players such that there then exist an SPNE of the original game and an SPNE of the modified game that yield the same equilibrium payoffs? If yes, indicate which payoff can be changed and show the solution. If no, argue why this is.

Solution: Consider the original (imperfect information) game. Change player 1's payoff after (A, r, L') from 10 to 0. Then it will be optimal for him to play L and R'. Imposing this, the game can be rewritten as the following bimatrix:

Player 2

		l	r
Player 1	A	$\underline{4}, \underline{2}$	$1, \underline{2}$
	B	$1, 2$	$\underline{5}, \underline{3}$

The two pure-strategy SPNE are therefore (ALR', l) and (BLR', r). The SPNE in the modified (perfect information) game are now (BLR', lr') and (BLR', rr'). The equilibrium payoffs are the same in the second SPNE of the original game, and in the SPNEs of the modified game.

2. Consider the signaling game shown in Figure 2 below.

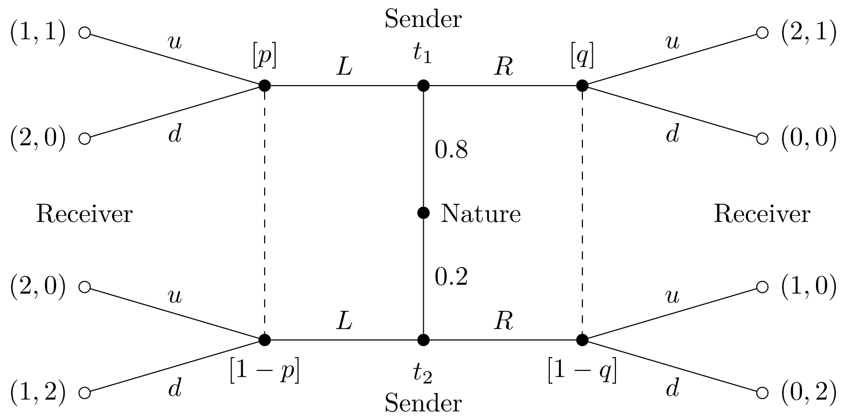


FIGURE 2: SIGNALING GAME

(a) Show that there exists a pooling perfect Bayesian equilibrium (PBE) in which both Sender types play L. Be careful to specify the beliefs p and q that support this equilibrium.

Solution: First, $m(t_1) = m(t_2) = L$ implies that $p = 0.8$. On the other hand, q cannot be calculated using Bayes' Rule. Then,

$$u_R(u, L; p) = 0.8 \cdot (1) + 0.2 \cdot (0) = 0.8;$$
$$u_R(d, L; p) = 0.8 \cdot (0) + 0.2 \cdot (2) = 0.4.$$

It follows that $a^*(L) = u$. Thus

$$u_S(L, a^*(L); t_1) = 1;$$
$$u_S(L, a^*(L); t_2) = 2.$$

Notice that t_2 will never deviate from this, as his highest payoff from playing R is 1. Hence, $m^*(t_2) = L$. However, for t_1, playing L is only a best response if $a^*(R) = d$. To assure this, we need that

$$\begin{aligned} u_R(d, R; q) &\geq u_R(u, R; q) \\ \Leftrightarrow q \cdot (0) + (1-q) \cdot (2) &\geq q \cdot (1) + (1-q) \cdot (0) \\ \Leftrightarrow 2(1-q) &\geq q \\ \Leftrightarrow q &\leq \frac{2}{3}. \end{aligned}$$

The pooling equilibrium is then $(LL, ud; p = 0.8, q \leq \frac{2}{3})$.
(Here, we have used the following convention for the notation: The first letter of the sender's strategy is the message of t_1, the second letter is the message of t_2. The first letter of the receiver's strategy is the action after L, the second is the action after R.)

(b) Does this pooling PBE satisfy SR5 and SR6?

Solution: There are no strictly dominated strategies for the Sender, and therefore the equilibrium satisfies SR5. Notice that R is equilibrium dominated for t_2 who gets 2 in equilibrium, but at most 1 from deviation to R. However, it is not equilibrium dominated for t_1, who gets 1 in equilibrium, but who can, potentially, get 2 from deviating. Therefore, SR6 implies $q = 1$. Since we require $q \leq 2/3$ for the pooling equilibrium to exist, there is no pooling equilibrium. Hence, the equilibrium does not satisfy SR6.

(c) Are there any separating PBE? If yes, show that one such equilibrium exists. If no, demonstrate that no such equilibrium exists.

Solution: First, let us look for a separating equilibrium with $m(t_1) = L$ and $m(t_2) = R$. This yields $p = 1$ and $q = 0$. Then, $a^*(L) = u$ and $a^*(R) = d$. In this case, t_1 gets 1 in equilibrium and 0 if he deviates, so $m^*(t_1) = L$. However, type t_2 gets 0 in equilibrium and 2 if he deviates. This cannot be an equilibrium.

Second, let us look for a separating equilibrium with $m(t_1) = R$ and $m(t_2) = L$. This yields $p = 0$ and $q = 1$. Then, $a^*(L) = d$ and $a^*(R) = u$. In this case, t_1 gets 2 in equilibrium and 2 if he deviates, so $m^*(t_1) = R$. Type t_2 gets 1 in equilibrium, and 1 if he deviates. Thus, $m^*(t_2) = L$.

In conclusion, there exists a pure-strategy separating equilibrium which is $(RL, du; p = 0, q = 1)$.

3. Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type η, which measures their ability. There are two worker types: $\eta \in \{L, H\}$. Nature chooses the worker's type, with $p = \mathbb{P}(H)$. The worker observes his own type, but the firm does not.

   The productivity y of the worker depends only on his type: $y(\eta, e) = \theta_\eta$. Education is thus non-productive. Assume that $\theta_H = 2$ and $\theta_L = 1$.

   The worker can choose his level of education: $e \in \mathbb{R}^+$. The cost to him of acquiring education is

   $$c(e, \eta) = \frac{\sqrt{e}}{\theta_\eta}.$$

   Education is observed by the firm, who then forms beliefs about the worker's type: $\mu(e) = \mathbb{P}(H|e)$. We assume that the firm is in competition such that it pays the expected productivity:

   $$w(e) = \mathbb{E}(\theta_\eta|e),$$

   where the expectation is calculated given the firm's beliefs μ about the worker's type.

   The payoff to a worker conditional on his wage, type and education is

   $$u(w, \eta, e) = w - c(e, \eta).$$

   We will look for pure-strategy perfect Bayesian equilibria (PBE). Denote the equilibrium level of education chosen by the two types, respectively, by e^*_H and e^*_L.

   (a) First, we will look for a separating equilibrium in which $e^*_H = 1$ and $e^*_L = 0$. Throughout this part, you can assume that the off-equilibrium-path beliefs are $\mu^*(e) = 0$ if $e \neq e^*_L, e^*_H$.

      (i). Specify the beliefs that must apply on the equilibrium path.

      (ii). Then argue that given the beliefs, the worker should only ever choose either $e = 0$ or $e = 1$.

   Solution: On the equilibrium path, $\mu^*(0) = 0$ and $\mu^*(1) = 1$. The beliefs are thus

   $$\mu^*(e) = \begin{cases} 1 \text{ if } e = 1, \\ 0 \text{ otherwise.} \end{cases}$$

   This implies that the wage schedule becomes

   $$w^*(e) = \begin{cases} 2 \text{ if } e = 1, \\ 1 \text{ otherwise.} \end{cases}$$

   For $e \neq 0, 1$ we have $u(w^*(e), \eta, e) < u(1, \eta, 0)$, and therefore the worker should never choose $e \neq 0, 1$, since it is dominated by choosing $e = 0$.

   (b) Using your answer to the previous question, show that there is a separating PBE where $e^*_H = 1$ and $e^*_L = 0$. Be sure to fully specify beliefs and equilibrium strategies.

Solution: Given the answer to (a), the two optimality conditions for equilibrium are

$$u(1, L, 0) \geq u(2, L, 1) \Leftrightarrow 1 - 0 \geq 2 - \frac{\sqrt{1}}{1} \Leftrightarrow 1 \geq 1.$$
$$u(2, H, 1) \geq u(1, H, 0) \Leftrightarrow 2 - \frac{\sqrt{1}}{2} \geq 1 - 0 \Leftrightarrow \frac{3}{2} \geq 1.$$

Thus, since both conditions hold, there is a PBE with $(e_L^* = 0, e_H^* = 1; \mu^*)$.

(c) Continue to assume that the off-equilibrium-path beliefs are $\mu^*(e) = 0$ if $e \neq e_L^*, e_H^*$. Also, continue to consider $e_L^* = 0$. Find all the values of e_H^* such that a separating PBE exists. Be sure to fully specify beliefs and equilibrium strategies.

Solution: Repeating the argument of (a), it will only be optimal to choose either $e = 0$ or $e = e_H^*$. The two optimality conditions are then

$$u(1, L, 0) \geq u(2, L, e_H^*) \Leftrightarrow 1 - 0 \geq 2 - \frac{\sqrt{e_H^*}}{1} \Leftrightarrow e_H^* \geq 1.$$
$$u(2, H, e_H^*) \geq u(1, H, 0) \Leftrightarrow 2 - \frac{\sqrt{e_H^*}}{2} \geq 1 - 0 \Leftrightarrow e_H^* \leq 4.$$

For $e_H^* \in [1, 4]$ there is a separating PBE with $(e_L^* = 0, e_H^*; \mu^*)$, where $\mu^*(e_H^*) = 1$ and $\mu^*(e) = 0$ for $e \neq e_H^*$.

(d) Now apply SR6 (equilibrium dominance). Which of the equilibria you found in (c) satisfy SR6?

Solution: Start with H:

$$u(2, H, e_H^*) > u(2, H, e) \Leftrightarrow 2 - \frac{\sqrt{e_H^*}}{2} > 2 - \frac{\sqrt{e}}{2} \Leftrightarrow e > e_H^*.$$

Then L:

$$u(1, L, 0) > u(2, L, e) \Leftrightarrow 1 - 0 > 2 - \frac{\sqrt{e}}{1} \Leftrightarrow e > 1.$$

Hence, $e \in (1, e_H^*]$ is equilibrium dominated for L but not for H. The updated beliefs are

$$\mu^{**}(e) = \begin{cases} 1 \text{ if } e = e_H^* \\ 1 \text{ if } e \in (1, e_H^*) \\ 0 \text{ otherwise.} \end{cases}$$

Thus, the wage becomes

$$w^{**}(e) = \begin{cases} 2 \text{ if } e = e_H^* \\ 2 \text{ if } e \in (1, e_H^*) \\ 1 \text{ otherwise.} \end{cases}$$

Therefore, if $e_H^* \in (1, 4]$, then H will deviate to some $e \in (1, e_H^*)$, since this yields the same wage for less education. This is, however, not possible if $e_H^* = 1$. It follows that only the equilibrium with $e_H^* = 1$ satisfies SR6.

(e) Comment on SR6. Do you think it is a reasonable requirement? Explain your answer.

Solution: Ad lib.